

Price Twenty-Five Cents.

VOL. I.

No. XI.

THE

MATHEMATICAL MONTHLY.

AUGUST, 1859.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

CAMBRIDGE:

PUBLISHED BY JOHN BARTLETT.

LONDON:

TRÜBNER AND CO.

1859.

CONTENTS.

AUGUST, 1859.

	PAGE
PRIZE PROBLEMS FOR STUDENTS,	353
REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VII., Vol. I.	354
NOTES AND QUERIES,	361
ANOTHER SOLUTION OF PRIZE PROBLEM II., No. IV. By George Eastwood,	364
THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE. By W. Ferrell,	366
ON THE SOLUTION OF EQUATIONS. By John Borden,	373
EDITORIAL ITEMS,	383

TERMS.

A single Copy,	\$3.00 per annum.
Two Copies to one Address,	5.00 "
Five Copies " "	11.00 "
Ten Copies " "	20.00 "

Payable invariably in advance.

JOHN BARTLETT, *Publisher.*

Entered according to Act of Congress in the year 1859, by

J. D. RUNKLE,

In the Clerk's Office of the District Court of the District of Massachusetts.

CAMBRIDGEPORT:

W. F. Brown, Book and Job Printer, 421 Main Street.

THE
MATHEMATICAL MONTHLY.

Vol. I... AUGUST, 1859.... No. XI

PRIZE PROBLEMS FOR STUDENTS.

I.

Solve the equations

$$\begin{aligned}x + y &= a \\ (x^3 + y^3)(x^2 + y^2) &= b,\end{aligned}$$

and give a discussion of the values of the roots.

II.

Let A, B, C be the angles, and a, b, c the opposite sides, of a plane triangle; it is required from the relation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

to deduce the formula

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

III.

A number n of equal circles touch each other externally, and include an area of a square feet; to find the radii of the circles. (Communicated by ARTEMAS MARTIN, Esq.)

IV.

If the sides of a spherical trapezium be denoted by a, b, c, d , the

diagonals by δ_1 and δ_2 , and the distance between the middle points of the diagonal by A ; show that

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \frac{1}{2} \delta_1 \cos \frac{1}{2} \delta_2 \cos A.$$

(Communicated by GEORGE EASTWOOD, Esq.)

V.

From an urn containing four white and four black balls, four are repeatedly drawn and replaced. A agrees to pay B one dollar every time the four balls drawn are equally divided between white and black; but if three, or all four, are of the same color, B is to pay A one dollar. Who has the advantage, and what is its value for each drawing? (Communicated by SIMON NEWCOMB, Esq.)

The solution of these problems must be received by the first of October, 1859.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE
PRIZE PROBLEMS IN No. VII., Vol. I.

The first Prize is awarded to O. B. WHEELER, student in the University of Michigan, at Ann Arbor.

The second Prize is awarded to WILLIAM EGERTON, student in the Baltimore College, Baltimore, Maryland.

PRIZE SOLUTION OF PROBLEM I.

"On two sides AC and BC of any triangle ABC , let any parallelograms $ACDE$ and $BCFG$ be described. Let ED and FG produced, meet in H ; join HC , and through A and B draw AL and BM equal and parallel to HC . Join LM . It is required to prove that the parallelogram $ALMB$ on the side AB is equal to the sum of the parallelograms on the sides AC and BC .

"Show also that the Pythagorean proposition is a particular case of this proposition."—PAPPUS.

The parallelograms AH and AD are equivalent; so are the parallelograms BH and BF , since they have the same base and altitude. Therefore $LH = AC$, and $HM = CB$. But since AL and BM are each equal and parallel to HC , they are equal and parallel to each other, and $LM = AB$. Therefore the triangle LHM is equal to ABC , since they are mutually equilateral. Take away the triangle $OC P$ from each, and we have left $LHMP CO$, equivalent to $AOPB$. Add ALO and BMP to both, and we have

$$ALHC + BMHC = AEDC + BGFC = ALMB.$$

Or thus: Produce HC till it meets AB in I ; then will $AD = AH = AK$. For $AI = LK$, and the triangles $LHK = ACK$. Taking away the common part OCK , and adding ALO to both remainders, we have $AH = AD = AK$. Also $BF = BK$. $\therefore AD + BF = AM$.

COROLLARY. The parallelograms AK and BK have the same altitude, and are therefore to each other as their bases AI and BI . Hence $AD : BF :: AI : IB$; that is, the line HC produced cuts the third side AB into parts which are to each other as the parallelograms described on the adjacent sides.

COR. 2. If triangles be described on the sides AC and BC , and lines be drawn through their vertices Q and R parallel to the sides AC and BC , and produced till they meet in H , and the remaining lines be drawn as in the figure, then the vertex of the triangle described on AB , which shall be equivalent to the sum of the other two, will be found anywhere in the line LM ; for a triangle is equivalent to half the parallelogram having the same base and altitude.



Fig. 1.

COR. 3. The triangles AQC and BRC are to each other as $AI:BI$, since the triangles are the halves of AD and BF .

COR. 4. The triangle ASI is equivalent to AQC , and BSI to BRC .

COR. 5. When ACB is right-angled at C , and squares are de-

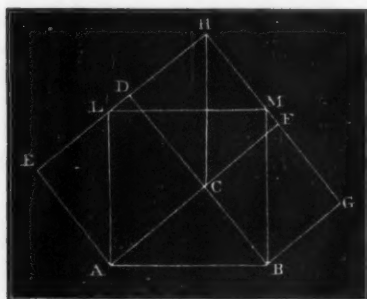


Fig. 2.

scribed on AC and BC , the diagonal HC equals AB . For the two triangles CDH and ACB have $AC=CD$, $BC=DH$, and C and D right angles. $\therefore CH=AB=AL$. Also the angle $DCH=CAB$, and $DHC=CBA=HCF=LAC$. $\therefore DCH+DHC=CAB+LAC =$ a right angle. $ALMB$ is there-

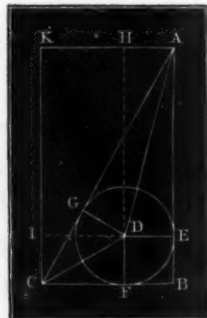
fore a square equivalent to the sum of the other two.

This solution is by DAVID TROWBRIDGE. In most of the other solutions, HC (Fig. 1) was produced to I ; then $AD=AH=AK$, and $BF=BH=BK$, as is easily seen. Therefore, by addition, $AD+BF=AM$.

PRIZE SOLUTION OF PROBLEM II.

"The area of a right-angled triangle is equivalent to the rectangle of the differences between the radius of the inscribed circle and the two shorter sides respectively; or the rectangle of the segments of the hypotenuse made by a perpendicular let fall upon it from the centre of the inscribed circle."

Let b be the base and a the altitude of the triangle ABC , and r the radius of the inscribed circle. It will be readily seen that the triangle $GDA=ADE=DAH$. $\therefore GDEA=EDHA$. $GDC=CDF=DCI$. $\therefore GDFC=FDIC$. $ACB=CDHAB=ar+(b-r)r$
 $=ar+br-r^2$.



$$KCA = ACB = ab - (ar + br - r^2) = ab - ar - br + r^2;$$

$$\text{but } KIDH = (a-r)(b-r) = ab - ar - br + r^2.$$

$$\therefore ABC = KIDH = (a-r)(b-r) = AE \times CF = AG \times CG.$$

This solution is by C. HERSCHEL.

PRIZE SOLUTION OF PROBLEM III.

"The distance between two points, A and B , is a miles. A person starts at A and travels the first day one m th his distance to B ; the second day he travels back one m th his distance to A ; the third day he turns and travels one m th his distance to B , and so on. How far will he travel in n days, and how far will he be from A ?"

Let a, a_1, a_2 , &c., denote his distances from the point A or B towards which he travels on the successive days; and let x_1, x_2, x_3 , &c., denote the distances travelled. Then, by the question

$$(1) \quad \frac{a}{m} = x_1, \quad \frac{a_1}{m} = x_2, \quad \frac{a_2}{m} = x_3, \quad \&c.,$$

and by addition

$$(2) \quad \frac{1}{m}(a + a_1 + a_2 + \dots + a_{n-1}) = x_1 + x_2 + x_3 + \dots + x_n = S_n$$

the whole distance travelled in n days. Next we have

$$\begin{aligned} a &= a, & m x_1 &= m x_1, & x_1 &= x_1 \\ a_1 &= a - a + x_1, & m x_2 &= m x_1 - m x + x_1, & x_2 &= x_1 - \frac{m-1}{m} x_1, \\ (3) \quad a_2 &= a - a_1 + x_2, & m x_3 &= m x_1 - m x_2 + x_2, & x_3 &= x_1 - \frac{m-1}{m} x_2, \\ &: &: &: &: &: \\ a_n &= a - a_{n-1} + x_n, & m x_{n+1} &= m x_1 - m x_n + x_n, & x_{n+1} &= x_1 - \frac{m-1}{m} x_n. \end{aligned}$$

By addition, the third set gives

$$S_n + x_{n+1} = (n+1)x_1 - \frac{m-1}{m} S_n;$$

hence, and by (1)

$$(4) \quad S_n = \frac{(n+1)m x_1 - m x_{n+1}}{2m-1} = \frac{(n+1)a - a_n}{2m-1}.$$

To find the value of $a_n = m x_{n+1}$, put $\frac{m-1}{m} = p$, and by succes-

sive substitutions, the last set of equations (3) give

$$(5) \quad x_1 = x_1, x_2 = x_1(1-p), x_3 = x_1(1-p+p^2), \text{ and so on to } \\ x_{n+1} = x_1(1-p+p^2-p^3+p^4-\dots \pm p^n).$$

The upper sign is to be used when n is *even*, and the lower when it is *odd*. Summing the series (5) becomes

$$(6) \quad x_{n+1} = x_1 \frac{1 \pm p^{n+1}}{1+p}, \text{ and } m x_{n+1} = a_n = a \frac{1 \pm p^{n+1}}{1+p};$$

or, replacing the value of p ,

$$(7) \quad a_n = m x_{1+n} = a \frac{m^{n+1} \pm (m-1)^{n+1}}{m^n(2m-1)}.$$

Substituting this value of a_n in (4) and reducing, we get

$$(8) \quad S_n = \frac{(n+1)a}{2m-1} - \frac{m^{1+n} \pm (m-1)^{n+1}}{m^n(2m-1)^2}.$$

When n is odd, (7) denotes the distance from A ; but when n is even, this distance is $a - a_n$. This solution is by DAVID TROWBRIDGE.

PRIZE SOLUTION OF PROBLEM IV.

"The volume of any right cone equals the product of its whole surface by one third the radius of the inscribed sphere." — Communicated by Prof. SNELL.

Let a sphere be inscribed in the right cone. Circumscribe a regular polygon about the base of the cone, and join the vertices of the polygon with the vertex of the cone. The faces of this circumscribed pyramid will be tangents to both the cone and sphere. Join the vertices of the solid angles of the pyramid and the centre of the sphere, and thus divide it into as many other pyramids as it has faces. All these pyramids have a common vertex at the centre of the sphere, and their common altitude is the radius of the sphere, since all their bases are tangents to it. The solidity of each pyramid is the product of its base by one third of the radius of the sphere, and therefore the solidity of the pyramid circumscribing the cone is the product of its whole surface by one third of the radius of the sphere. Let, now,

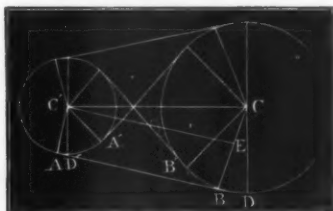
the number of sides of the polygon circumscribing the cone's base be continually increased; the limit of the polygon is the base of the cone; and the limit of the whole surface of the pyramid is the whole surface of the cone. Hence the volume of the cone is the product of its whole surface by one third of the radius of the inscribed sphere.

This solution is by J. C. ELLIOTT; and the same reasoning was used by O. B. WHEELER.

PRIZE SOLUTION OF PROBLEM V.

"One pulley drives another by means of a belt; give the length of the belt l , the diameter D , of the larger pulley, the distance a between the centres of the pulleys; to find the diameter d of the smaller pulley. Find also a simple approximate formula for the use of machinists."

Let $CB = R = \frac{1}{2}D$, and $C'A = r = \frac{1}{2}d$. If b , D , and a remain the same, it is evident that d will have different values according as the belt is a crossed or open one. For a crossed belt the sum of the straight parts is $2\sqrt{a^2 - (R+r)^2}$, since each is a leg of a right-triangle, having a for the hypotenuse and $R+r$ for the other leg. In this triangle the angle opposite $R+r$ equals $B'CD$ or $A'C'A'$, or $\sin^{-1} \frac{R+r}{a}$, and therefore the arc DBB' equals $R \sin^{-1} \frac{R+r}{a}$. The curved part of the belt on the greater pulley is then $R\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right)$, and on the smaller pulley it is $r\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right)$.



$$\therefore l = 2\sqrt{a^2 - (R+r)^2} + (R+r)\left(\pi + 2 \sin^{-1} \frac{R+r}{a}\right).$$

From this equation r can be found approximately. For an open belt, AB or $C'E$ is $\sqrt{a^2 - (R-r)^2}$. The angle $B'CD = C'C'E$, the

sine of which is $\frac{R-r}{a}$. The arc BD equals $R \sin^{-1} \frac{R-r}{a}$, and the arc AD' equals $r \sin^{-1} \frac{R-r}{a}$.

$$\begin{aligned} \therefore l &= 2\sqrt{a^2 - (R-r)^2} + R\left(\pi + 2\sin^{-1} \frac{R-r}{a}\right) + r\left(\pi - 2\sin^{-1} \frac{R-r}{a}\right) \\ &= 2\sqrt{a^2 - (R-r)^2} + (R+r)\pi + 2(R-r)\sin^{-1} \frac{R-r}{a}. \end{aligned}$$

From this equation another approximate value of r can be found.

Since $R-r$ is small compared with a , we have approximately,

$$2\sqrt{a^2 - (R-r)^2} = 2a - \frac{(R-r)^2}{a}, \quad \sin^{-1} \frac{R-r}{a} = \frac{R-r}{a}.$$

Therefore, for an open belt the approximate formula is

$$\begin{aligned} &= 2a - \frac{(R-r)^2}{a} + (R+r)\pi + \frac{2(R-r)^2}{a}, \\ &= 2a + \frac{(R-r)^2}{a} + (R+r)\pi, \end{aligned}$$

from which

$$2r = d = 2R - a\pi \pm \sqrt{8(a^2 + r^2) - l a + a^2 \pi^2}.$$

This solution is by O. B. WHEELER.

SOLUTION 2d. If the angle $CC'E = BCD$ be denoted by φ , then $2AB = 2a \cos \varphi$, $BE = R - r = a \sin \varphi$, $BD = R\varphi$, $AD' = r\varphi$.

$$\begin{aligned} \therefore l &= 2a \cos \varphi + R\pi + 2R\varphi + r\pi - 2r\varphi \\ &= 2a \cos \varphi + (R+r)\pi + 2(R-r)\varphi \\ &= 2a \cos \varphi + (R+r)\pi + 2a\varphi \sin \varphi \\ (1) \quad &= 2a \cos \varphi + 2a\varphi \sin \varphi + 2R\pi - a\pi \sin \varphi. \end{aligned}$$

$$\text{But} \quad \cos \varphi = 1 - \frac{1}{2}\varphi^2 + \frac{1}{1.2.3.4}\varphi^4 - \&c.,$$

$$\sin \varphi = \varphi - \frac{1}{1.2.3}\varphi^3 + \frac{1}{1.2.3.4.5}\varphi^5 - \&c.;$$

and substituting these values of $\cos \varphi$ and $\sin \varphi$ in (1) we shall find, after reversing the series, that

$$\varphi = -\frac{2}{\pi} \left(\frac{l - 2R\pi - 2a}{2a} \right) + \frac{4}{\pi^3} \left(\frac{l - 2R\pi - 2a}{2a} \right)^3 - \&c.$$

$$\begin{aligned}\text{But } 2r = d = 2R - 2a \sin \varphi \\ = D - 2a \left(\varphi - \frac{1}{1.2.3} \varphi^3 + \frac{1}{1.2.3.4.5} \varphi^5 - \&c. \right).\end{aligned}$$

For a simple approximate formula, let $\cos \varphi = 1$, and $\sin \varphi = \varphi$, and (1) becomes

$$\begin{aligned}l &= 2a + 2a\varphi^2 + 2R\pi - a\pi\varphi, \\ \therefore \varphi &= \frac{1}{4}\pi \pm \sqrt{\frac{1}{16}\pi^2 + \frac{l-2R\pi-2a}{2a}}, \\ \therefore d &= D - 2 \left(\frac{1}{4}\pi \pm \sqrt{\frac{1}{16}\pi^2 + \frac{l-2R\pi-2a}{2a}} \right).\end{aligned}$$

This solution is by ASHER B. EVANS.

JOSEPH WINLOCK.
CHAUNCEY WRIGHT.
TRUMAN HENRY SAFFORD.

NOTES AND QUERIES.

1. *Subtraction of Fractions.* Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions; then, as is easily seen,

$$\frac{a}{b} - \frac{c}{d} = \frac{a(d-c) - c(b-a)}{bd} = \frac{ad-bc}{bd}.$$

When $d-c$ is less than d , and $b-a$ is less than b , the numerator of the second form will be most readily computed; and the advantage will be great when the terms of the fractions are large numbers, but nearly equal to each other. TERQUEM'S *Annales de Mathematique*.

2. It is interesting to notice, that the demonstration of the Pythagorean proposition on page 231 of the MATHEMATICAL MONTHLY is essentially the same as the Indian demonstration contained in the *Bija Ganita*, and referred to as the "figure of the bride's chair," &c. It is involved in the square AK (fig. p. 231). For let a represent the side of the square, and b and c the legs of the equal

right-angled triangles. Then the area of the four equal triangles is $2bc$; and $(b-c)^2$ is the area of the square SQ . *Whence $a^2 = 2bc + (b-c)^2 = b^2 + c^2$. This demonstration is given by Dr. HUTTON (Tracts, London, 1812, 3 Vols. 8vo), in his History of Algebra, where may be found a very complete description of the *Bija Ganita* and *Lilawati*. Dr. HUTTON had in his possession Persian MSS. containing translations of both these works. — H. W. RICHARDSON, Waterville College, Maine.

3. *Problem.* A man left 17 horses to be divided among his three sons, the first to have $\frac{1}{2}$, the second $\frac{1}{3}$, and the youngest $\frac{1}{4}$ of the number. They could not agree as to the division, because it required some of the horses to be divided; thus, $\frac{1}{2}$ of 17 = $8\frac{1}{2}$, $\frac{1}{3}$ of 17 = $5\frac{2}{3}$, $\frac{1}{4}$ of 17 = $4\frac{1}{4}$. The sum of the three shares was $16\frac{1}{4}$ horses, and the whole number was not distributed. They carried the case to a judge, who told them, that, if they would abide by his decision, he would give each more than his share, and each should have a whole number of horses. Accordingly, he brought his own horse from the stall, and put him with the 17 others. He then gave $\frac{1}{2}$ of 18 = 9 horses to the first; $\frac{1}{3}$ of 18 = 6 to the second; and $\frac{1}{4}$ of 18 = 4 to the third, making 17 in all. He then returned his own horse to the stall, and left the sons well satisfied. Was the decision just? * * *

4. A correspondent sends us the equation $2\pi\sqrt{-1} = 0$, and proves it as follows. In exponentials,

* This demonstration has been sent us by JOHN M. BROWN, Esq., of Frankfort, Ky. We have also received several others, which are filed for publication. Prof. ALPHEUS CROSBY, Principal of the Normal School at Salem, Mass., in his work on Geometry, refers to a *Treatise on the Pythagorean Proposition* by HOFFMAN, published at Mayence in 1819, which contains thirty-three different demonstrations of this celebrated theorem. We intend, as soon as we can get a copy of the work, to give in the MONTHLY a brief outline of each demonstration. But if any of our readers have the work, we shall be much obliged for such a synopsis as we propose.

$$\sin x = \frac{1}{2\sqrt{-1}} (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}).$$

Make $x = \pi$; then $\sin \pi = 0 = \frac{1}{2\sqrt{-1}} (e^{\pi\sqrt{-1}} - e^{-\pi\sqrt{-1}})$; or $e^{\pi\sqrt{-1}} = e^{-\pi\sqrt{-1}}$. Multiplying both members by $e^{-\pi\sqrt{-1}}$ we get $e^{2\pi\sqrt{-1}} = e^0 = 1$. $\therefore 2\pi\sqrt{-1} = 0$.

The last conclusion is not correct; for if it were, some one of the factors of $2\pi\sqrt{-1}$ must be zero, which is not the case.

Make $x = n\pi$; then $e^{n\pi\sqrt{-1}} = e^{-n\pi\sqrt{-1}} = \frac{1}{e^{n\pi\sqrt{-1}}}$. Therefore, $e^{2n\pi\sqrt{-1}} = 1$. $\therefore \log 1 = 2n\pi\sqrt{-1}$. And since this is true for all values of n , it follows that $\log 1 = 0$, for $n = 0$, but is imaginary for all other values of n ; that is, $\log 1$ has an infinite number of values, only one of which is real.

It is true in general, that any positive number has one real, and an infinite number of imaginary, logarithms. Let x be the Napierian logarithm of N . Then $Ne^{2n\pi\sqrt{-1}} = N = e^x$. Therefore, $\log N + 2n\pi\sqrt{-1} = x$; or $\log N = x - 2n\pi\sqrt{-1}$, which is real only when $n = 0$.

5. *The least Common Multiple.* Some of the arithmetics say, divide the given numbers by *any* number which will divide two or more of them without a remainder, next divide the quotients and undivided numbers by *any* number, &c.; others say, divide by *any prime* number. Will both rules give the correct answer?

6. What is the origin of the term, *Pons Asinorum*, as applied to the fifth proposition of the first book of Euclid?

7. *Twenty-Two Systems of Coördinates.* The usual right-line, or Cartesian coördinates, are x, y ; the polar r, φ ; the directions of the normal and tangent, or the angles they make with an assumed axis, are ν and τ ; s is the length of the curve; ρ is its radius

of curvature; ε the angle between the radius vector and tangent.

$$(1) y = F(x) . (2) r = F(\varphi) . (3) \varrho = F(r) . (4) \tau = F(s) .$$

$$(5) \varrho = F(s) . (6) \varepsilon = F(\varphi) . (7) \tau = F(\varphi) . (8) r = F(x) .$$

$$(9) x = F(\varphi) . (10) x = F(\varepsilon) . (11) \tau = F(x) . (12) \varrho = F(x) .$$

$$(13) x = F(s) . (14) \varepsilon = F(r) . (15) \tau = F(r) . (16) \varrho = F(r) .$$

$$(17) r = F(s) . (18) \varrho = F(\varphi) . (19) \varphi = F(s) . (20) r = F(\varepsilon) .$$

$$(21) \varrho = F(\varepsilon) . (22) \varepsilon = F(s) .$$

The student will find it an excellent exercise to express some familiar curve in as many of these systems as possible. [See Paper on this subject by Rev. THOMAS HILL, in Proc. Am. As. Ad. Sc. 12th meeting, p. 1.]

ANOTHER SOLUTION OF PRIZE PROBLEM II., No. IV.

By GEORGE EASTWOOD, Saxonville, Mass.

I. In solving this problem, I shall use Professor GUDERMANN'S method of Spherical Rectangular Coördinates, on account of the remarkable analogy which it exhibits between the properties of lines drawn on the surface of a sphere, and those of lines drawn in a plane. I shall assume, as Mr. MERRILL does in his solution, (No. VIII.), that ABC is the proposed triangle, AB the given base, P its pole, PO a prime meridian passing through the middle of the base, and PCD another meridian passing through the vertex C ; that the side AC intersects PO in E , and that BC meets it in F .

Let the vertex C be projected on PO in the point G , and put $AO = OB = \alpha$, $OE = \beta$, $OF = \beta'$, $OD = x$, and $OG = y$. Then, if we agree to represent the trigonometric tangents of the coördi-

nate arcs, by the symbols of those arcs the equation of the side AC will be defined by

$$(1) \quad y = \frac{\beta}{\alpha} x + \beta,$$

and of BC by

$$(2) \quad y = -\frac{\beta'}{\alpha} x + \beta'.$$

By spherics,

$$\tan A = \frac{\beta \sec \alpha}{\alpha},$$

$$(3) \quad = \frac{y \sec \alpha}{\alpha + x}, \text{ by reason (1).}$$

$$\tan B = \frac{\beta' \sec \alpha}{\alpha},$$

$$(4) \quad = \frac{y \sec \alpha}{\alpha - x}, \text{ by reason of (2).}$$

$$\text{But} \quad \frac{(3)}{(4)} = \frac{\alpha - x}{\alpha + x}$$

is, by the question, a given ratio $= m$, suppose.

$$(5) \quad \therefore x = \frac{(1-m)\alpha}{1-m};$$

that is, the vertex of the triangle is always on the meridian circle PCD .

From (5) we have, since x and α are tangents,

$$\sin x = \frac{\pm (1-m) \sin \alpha}{[(m+1)^2 \cos^2 \alpha + (1-m)^2 \sin^2 \alpha]^{\frac{1}{2}}},$$

$$\cos x = \frac{\pm (m+1) \cos \alpha}{[(m+1)^2 \cos^2 \alpha + (1-m)^2 \sin^2 \alpha]^{\frac{1}{2}}}.$$

$$\begin{aligned} \text{Hence} \quad \sin AD &= \sin(\alpha + x), \\ &= \frac{\sin 2\alpha}{(m^2 + 2m \cos 2\alpha + 1)^{\frac{1}{2}}}. \end{aligned}$$

If in this equation we make $2\alpha = a$, we shall have exactly the same result that Mr. MERRILL obtains in his solution, page 259.

II. By the same method of coördinates we shall find that Prize Problem II., No. V., is susceptible of a very neat and simple solution. For, if we designate the required coördinates by x and y , the given

points by $x' y'$ and $x'' y''$, and the intercepts of the axes of reference by α and β ; then, by known properties* of great circles arcs, we have

$$(1) \quad x = -\frac{1}{\alpha}, \quad y = -\frac{1}{\beta},$$

$$(2) \quad y' = -\frac{\beta}{\alpha} x' + \beta,$$

$$(3) \quad y'' = -\frac{\beta}{\alpha} x'' + \beta.$$

The elimination of α and β from (1), by means of (2) and (3), will obviously satisfy the required conditions of the problem. But the elegant solution of the problem by Mr. OSBORNE in the last MONTHLY, page 292, by an entirely different method, would seem to render further remarks unnecessary.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 307.]

SECTION IV.

ON THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

33. By the general motions of the atmosphere are meant all those motions produced by a difference of density between the equatorial and polar regions arising principally from a difference of temperature. If the motions of the atmosphere were not resisted by the earth's surface, the results of the preceding sections could be at once applied to them without any modifications, and hence towards the poles there would be a very rapid motion eastward, and in the equatorial

* These properties, and many other analogous ones, of great circle arcs, it is proposed to investigate in subsequent numbers of the MONTHLY.

regions towards the west, and the atmosphere would entirely recede from the poles, and be also depressed about 4,000 feet at the equator, as has been shown in section (2). Although the preceding results, when applied to the atmosphere, are very much modified by the resistances of the earth's surface, yet they will be of great advantage in explaining its general motions; for as there can be no resistance until there is motion, the atmosphere must have a tendency to assume, in some measure, the same motions and figure as in the case of no resistances. Hence, towards the poles the general motions of the atmosphere must be towards the east, and in the torrid zone towards the west; but as these motions, in consequence of the resistances, are small in comparison with those in the case of no resistances, instead of the atmosphere's receding entirely from the poles, as represented in Fig. 1, page 215, there must be only a comparatively small depression there, as represented in Fig. 5, and instead of its being about 4,000 feet lower at the equator than at

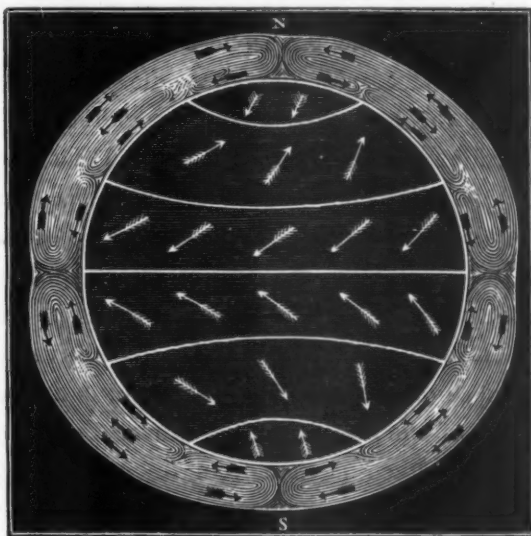


Fig. 5.

the place of its maximum height near the tropics (§ 18), there must be only a very slight depression there.

34. The force which overcomes the resistance of the earth's surface to the east and the west motions of the atmosphere depends upon the term in the least of our general equations (13) containing

D, θ as a factor, which depends upon the interchanging motion of the fluid between the equatorial and the polar regions, and hence the term must vanish at the equator and the poles. All the east or west motion of the atmosphere is consequently destroyed by the resistances at these places, and hence as D, θ vanishes there also, there is a belt of calms at the equator, called the equatorial calm belt, and there must be also a region of calms about the poles.

35. As the motion of the atmosphere is east towards the poles and west near the equator, somewhere between the equator and the poles there must be a parallel of no motion east or west, which, in the case of no resistance, was determined upon the hypothesis of an initial state of rest, and found to be at the parallel of 35° , § (18). In the case of the atmosphere this parallel is entirely independent of the initial state of the atmosphere, and depends in a great measure upon the law of resistance, and hence it cannot be accurately determined. It is evident, however, that the east and west motions of the atmosphere at the earth's surface must be such that the sum of the resistances of each part of the earth's surface multiplied into its distance from the axis of rotation, must be equal 0, else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere. Now, as the part of the earth's surface where the motion of the atmosphere is west is much farther from the axis than the part where it is east, the latter part must comprise more than half of the earth's surface, unless the velocity of the eastern motion towards the poles is much greater than that of the western motion near the equator. Therefore, since one-half of the earth's surface is contained between the parallels of 30° , the parallels of no east or west motion at the earth's surface must fall within these parallels, and they are accordingly found to be near the tropics, on the ocean. Hence the maximum height

of the atmosphere, as represented in Fig. (5), must also be near the same parallels.

36. The increase of pressure arising from the accumulation of atmosphere near the tropics, caused principally by the deflecting forces (§ 32) arising from the more rapid east and west motions of the atmosphere in the upper regions, where there is least resistance, gives the atmosphere a tendency to flow from beneath this accumulation both towards the equator and the poles, since the motions, and consequently the forces, which cause this accumulation, are much less near the surface. But on account of the greater density of the atmosphere towards the poles, it has a tendency also to flow, at the earth's surface, from the poles towards the equator. Between the parallels of greatest pressure and the equator, these tendencies combine, and produce a strong surface current, which, combining with the westward motion there, gives rise to the well-known north-east wind in the northern hemisphere, and the south-east wind in the southern hemisphere, called the trade winds. But between the parallels of greatest pressure and the poles, these tendencies are opposed to each other, and the one arising from the accumulation of atmosphere near the tropics being the greater in the middle latitudes, causes the atmosphere to flow at the earth's surface towards the poles; and this motion, combining with the general eastward motion of the atmosphere in those latitudes, gives rise to the south-west wind in the northern hemisphere and the north-west wind in the southern hemisphere, called the passage winds.

37. Near the poles, the tendency to flow towards the equator seems to be the greater, and causes a current there *from* the poles, which, being deflected westward (§ 32), causes a slight north-east wind in the north frigid zone, and a south-east wind in the south frigid zone. But this is only near the earth's surface; and the gener-

al tendency of the atmosphere in the upper regions must be towards the east, as will be seen.

38. Since the atmosphere near the tropics can have no motion in any direction at the earth's surface, there are calm belts there, called the tropical calm belts. Near the polar circles, where the polar and passage winds meet, there must also be calm belts, which may be called polar calm belts. The motions of the atmosphere, therefore, at the earth's surface, if they were not modified by the influence of continents, would be as represented in the interior of Fig. (5), in which the heavy lines represent the calm belts. On account of the influence of the continents, these belts are somewhat displaced and irregular, and on account of the varying position of the Sun, they change their positions a little in different seasons of the year.

The southern limit of the polar winds in the northern hemisphere, and also the limit between the trade and passage winds, has been determined by Prof. J. H. COFFIN, from the discussion of a great number of observations at different points, and given in a chart, in his treatise on the winds, published in the seventh volume of the Smithsonian Contributions.

39. That the atmosphere is depressed at the equator and the poles, and has its maximum height near the tropics, as has been represented, is indicated by barometrical pressure. It was formerly thought that this pressure, at the level of the ocean, was very nearly 30 inches in all latitudes; but it is now well established that it is much less towards the poles than near the tropics, and also a little less at the equator. Says Captain WILKES: "The most remarkable phenomenon which our observations have shown is the irregular outline of the atmosphere surrounding the earth as indicated by the pressure upon the measured column at different parts of the surface. Our barometrical observations show a depression

within the tropics, a bulging in the temperate zone, again undergoing a depression on advancing towards the arctic and antarctic circles." The mean of all the observations, as given in the Report of the Exploring Expedition, from Cape Henry to Madeira, taken between the parallels of 28° and 32° , was 31.215 inches; at Maderia, latitude $32^{\circ} 53'$, 30.176 inches; and in the rainy belt between the parallels of 8° and 12° , 29.987 inches. After passing the equator there was a slight elevation, again reaching its maximum near the tropic of Capricorn. Beyond this there was a gradual depression until about the parallel of 55° , where the barometer was rapidly depressed below 29 inches. After doubling Cape Horn and proceeding towards the equator, the height of the barometer gradually increased again to its usual height in the middle and equatorial latitudes. On sailing south again, in the Pacific Ocean, a depression of the barometer was again observed. The mean of all the observations taken on 22 days, in sailing from Callao to Tahiti, between the parallels of 10° and 15° , was 30.109 inches; and of those made on 32 days, between the parallels of 15° and 20° , was 30.147 inches. The mean of the observations made on 5 days, after leaving Sydney, between the parallels of 35° and 45° , was 30.305 inches; of those made on 7 days, between the parallels of 45° and 55° , was 29.790 inches; of those taken on 8 days, between the parallels of 55° and 65° , was 29.378 inches. The mean also of all those taken along the antarctic continent was 29.040 inches.

40. Says Sir JAMES ROSS (*Voyage to the Southern Seas*, Vol. 2, p. 383): "Our barometrical experiments appear to prove that the atmospheric pressure is considerably less at the equator than near the tropics; and to the south of the tropic of Capricorn, where it is greatest, a gradual diminution occurs as the latitude is increased, as will be shown from the following Table, derived from hourly observations of the height of the column of mer-

cury between the 20th of November, 1839, and the 31st of July, 1843."

EXTRACT FROM ROSS'S TABLE.

LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.
	inches.		inches.		inches.
Equator,	29.974	42° 53'	29.950	55° 52'	29.360
13° 0' S.	30.016	45 0	29.664	60 0	29.114
22 17	30.085	49 8	29.467	66 0	29.078
34 48	30.023	51 33	29.497	74 0	28.928
		54 26	29.347		

41. The following table, first published by M. SCHOUW, and reduced here from millimetres to English inches, shows that there is a similar bulging of the atmosphere in the middle latitudes, and depression at the pole in the northern hemisphere, as has been observed in the southern hemisphere.

PLACE.	LATITUDE.	PRESSURE.	PLACE.	LATITUDE.	PRESSURE.
		inches.			inches.
Cape,	33° 0' S.	30.040	London,	51° 30'	29.961
Rio Janeiro,	23 S.	30.073	Altona,	53 30	29.937
Christianburg,	5 30 N.	29.925	Dantzic,	54 30	29.925
La Guayra,	10	29.928	Konigsberg,	54 30	29.941
St. Thomas,	19	29.941	Apenrade,	55	29.905
Macao,	23	30.039	Edinburgh,	56	29.851
Teneriffe,	28	30.087	Christiana,	60	29.866
Madeira,	32 30	30.126	Bergen,	60	29.703
Tripoli,	33	30.213	Hardanger,	60	29.700
Palermo,	38	30.036	Reikiavig,	64	29.607
Naples,	41	30.012	Godthaab,	64	29.603
Florence,	43 30	29.996	Eyafjord,	66	29.669
Avignon,	44	30.000	Godhaven,	69	29.674
Bologna,	44 30	30.008	Upernavik,	73	29.732
Padua,	45	30.008	Mellville Isle,	74 30	29.807
Paris,	49	29.976	Spitzbergen,	75 30	29.795

42. From the preceding tables, it is seen that the barometrical pressure is much less, especially in the southern hemisphere towards the poles than at the equator, although the density towards the poles is much greater, and hence the depression there must be considerable.

ON THE SOLUTION OF EQUATIONS.

By JOHN BORDEN, Chicago, Illinois.

THE general equation of the second degree, $x^2 + Ax + B = 0$, may be solved as follows. Assume $x = -a + b$, $x = -a - b$ as the values of the roots. Then

$$(x + a - b)(x + a + b) = x^2 + 2ax + a^2 - b^2 = x^2 + Ax + B;$$

and therefore $A = 2a$, $B = a^2 - b^2$; or, $a = \frac{A}{2}$, $b = \pm \frac{1}{2} \sqrt{A^2 - 4B}$.

Therefore, $x = -\frac{A}{2} \pm \frac{1}{2} \sqrt{A^2 - 4B}$, as by the usual method.

The general form of the cubic equation is

$$(1) \quad x^3 + Ax^2 + Bx + C = 0.$$

If we suppose one root known, as $x = -c$, then the quotient obtained by dividing (1) by $x + c$ must equal zero; that is,

$$x^2 + (A - c)x - (A - c)c + B = 0.$$

Therefore the other two roots are

$$(2) \quad x = -\frac{A - c}{2} \pm \sqrt{A^2 + 2Ac - 4B - 3c^2}.$$

By making $x = x' - \frac{A}{3}$, as in the common transformation, (1) becomes

$$(3) \quad x'^3 + \left(B - \frac{A^2}{3}\right)x' + C - \frac{AB}{3} + \frac{2A^3}{27} = 0.$$

If, in this equation, $B - \frac{A^2}{3} = 0$, then

$$(4) \quad x' = \left(\frac{A^3}{27} - C\right)^{\frac{1}{3}}, \quad x' = -\frac{1}{2} \left(\frac{A^3}{27} - C\right)^{\frac{1}{3}} \pm \frac{1}{2} \left(\frac{A^3}{27} - C\right)^{\frac{1}{3}} \sqrt{-3},$$

If the third term of (3) reduces to zero, then

$$x' = 0, \quad x' = \pm \sqrt{\frac{A^2}{3} - B};$$

or lastly, if the third term, as well as the coefficient of x' , is zero, then all the values of x' are zero, and the values of x in (1) all equal $-\frac{A}{3}$. But, none of these cases occurring, (3) is of the general form of (1), if $A = 0$; hence if one root $= -c$, the expression for the other two becomes

$$(5) \quad x' = \frac{c}{2} \pm \sqrt{-4B' - 3c^2}.$$

Therefore, the general form of the roots of (3) are

$$(6) \quad x' = -c, \quad x' = \frac{c}{2} + d, \quad x' = \frac{c}{2} - d;$$

and those of (1), since $x = x' - \frac{A}{3}$, are

$$(7) \quad x = -\frac{A}{3} - c, \quad x = -\frac{A}{3} + \frac{c}{2} + d, \quad x = -\frac{A}{3} + \frac{c}{2} - d.$$

The difficulty, which arises in obtaining the values of c and d in terms of the coefficients appears to come from the fact, that any thing which can be predicated as true of one of the roots in general terms is also true of all the others.

There is one other transformation, of which an equation of the form $x^3 + Bx + C = 0$, for instance, is susceptible. If $x' = xy$, then $x'^3 + Bx'y^2 + Cy^3 = 0$, and we may assume $Bx'y^2 = Cy^3$; or $Bx'y^2 = E$; or $Cy^3 = d$; or, lastly, $x' = x, y = 1$, in which case $y = \frac{1}{x}$, and the equation becomes $y^3 + \frac{B}{C}y^2 + \frac{1}{C} = 0$, in which the roots, or values of y , are the reciprocals of the values of x , and the solution of one equation involves the solution of the other.

By combining equations (6) there results

$$(8) \quad x^3 - \left(\frac{3}{4}c^2 + d^2\right)x + \frac{c^3}{4} - cd^2 = 0.$$

From this it appears, that, unless d is imaginary, and $d^2 > \frac{3}{4}c^2$, the coefficient of x is essentially negative; and as the third term is the product of all the roots with their signs changed, and the two minor roots are of the same sign, and the maximum root of the contrary sign, that the third term and the maximum root will have contrary signs. If the coefficients of (8) be equated with those of

$$(9) \quad x^3 + Bx + C = 0,$$

and c be eliminated, the resulting equation is

$$(10) \quad d^6 + \frac{3}{2}Bd^4 + \frac{9}{16}B^2d^2 + \frac{1}{16}B^3 + \frac{27}{64}C^2 = 0; \text{ or}$$

$$(11) \quad d^3 + \frac{3}{4}Bd \pm \sqrt{-\frac{1}{16}B^3 - \frac{27}{64}C^2} = 0.$$

If the third term of (11) be equal to zero, then $d = 0$, and two of the roots of (8) are equal, as will appear from (6). But if $d = 0$, then the third term of (11) equals zero, and expresses the relative values of B and C in such case; namely, $4B^3 = -27C^2$. It also follows, as the third term of (8) equals the third term of (9), that $c = (-4C)^{\frac{1}{3}}$. And this is the value of the maximum root when the two minor roots are equal. Further, equation (10) shows that d may have six values, or two for each value of c in (6). And upon investigation, it will be found that d equals one half the algebraic difference of the roots of (9), taken two in a set. And since the two minor roots of (9) have the same sign, the values of d are equal one half their arithmetical difference, and at the same time the maximum root of (9) is equal to their arithmetical sum.

By reference to equation (10), it will appear, that, although it is of the sixth degree, yet in form it is of the third; and is composed of the two cognate factors represented in (11). And this leads to a consideration of the cognate factors into which any equation may be

decomposed. For, suppose the general cubic equation to be put under the form

$$(12) \quad (x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = x^6 - A'x^4 + B'x^2 - C' = 0,$$

and assume

$$(x^3 + Ax^2 + Bx + C)(x^3 - Ax^2 + Bx - C) = 0$$

as two of the cubic factors into which it may be decomposed. By multiplying we obtain

$$(13) \quad \begin{array}{l} x^6 - (A^2 - 2B)x^4 + (B^2 - 2AC)x^2 - C^2 = 0 = (12) \\ x^6 - \quad \quad \quad A'x^4 + \quad \quad \quad B'x^2 - C' = 0; \end{array}$$

and by equating the coefficients we have

$$(14) \quad A^2 - 2B = A', \quad B^2 - 2AC = B', \quad C^2 = C';$$

and by eliminating to find A we get

$$(15) \quad A^4 - 2A'A^2 - 8A\sqrt{C'} + A'^2 - 4B' = 0,$$

which is an equation of the fourth degree, wanting its second term.

But (12) is of the form

$$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = 0,$$

and equations (13) are respectively of the form

$$(x + a)(x + b)(x + c) = 0, \quad (x - a)(x - b)(x - c) = 0.$$

Therefore, A is equal to the sum of the square roots of the roots of (12), considered as an equation of the third degree. Therefore, the roots of an equation of the form of (15) are determined; and the bi-quadratics are solved. Or, by eliminating A from equations (14), we obtain

$$(16) \quad B^4 - 2B'B^2 - 8C'B + B'^2 - 4A'C' = 0,$$

which is also an equation of the fourth degree, wanting its second term, and this furnishes another solution of bi-quadratics. For B is equal to the sum of the products of the square roots of the roots or

(12) combined, two in a set, (12) being taken as an equation of the third degree.

By examining the cognate quadratic factors of an equation of the second degree, a relation of the like kind is established between equations of the second and third degrees, as follows :

$$(17) \quad (x^3 - a^3)(x^3 - b^3) = 0 = x^6 - A'x^3 + B',$$

$$(18) \quad (x - a)(x - b) = 0 = x^2 - Ax + B, \text{ the factor sought.}$$

$$(x^3 - a^3) = (x - a)\left(x + \frac{a}{2} \pm \frac{a}{2}\sqrt{-3}\right); \quad x^3 - b^3 = (x - b)\left(x + \frac{b}{2} \pm \frac{b}{2}\sqrt{-3}\right).$$

$$(19) \quad \begin{aligned} & \left(x + \frac{a}{2} + \frac{a}{2}\sqrt{-3}\right)\left(x + \frac{b}{2} + \frac{b}{2}\sqrt{-3}\right) \\ &= x^2 + \left(\frac{A}{2} + \frac{A}{2}\sqrt{-3}\right)x - \frac{B}{2} + \frac{B}{2}\sqrt{-3}. \end{aligned}$$

$$(20) \quad \begin{aligned} & \left(x + \frac{a}{2} - \frac{a}{2}\sqrt{-3}\right)\left(x + \frac{b}{2} - \frac{b}{2}\sqrt{-3}\right) \\ &= x^2 + \left(\frac{A}{2} - \frac{A}{2}\sqrt{-3}\right)x - \frac{B}{2} - \frac{B}{2}\sqrt{-3}. \end{aligned}$$

But the right-hand members of equations (18), (19), (20) multiplied together, give (17).

$$\therefore x^6 - (A^3 - 3AB)x^3 + B^3 = x^6 - A'x^3 + B' = 0.$$

(21) Whence $A^3 - 3AB = A'$, $B^3 = B'$, in which $a^3 + b^3 = A'$, $a^3b^3 = B'$, and $a + b = A$, $ab = B$. Whence

$$(22) \quad A = \left(\frac{A'}{2} + \frac{1}{2}\sqrt{A' - 4B'}\right)^{\frac{1}{3}} + \left(\frac{A'}{2} - \frac{1}{2}\sqrt{A' - 4B'}\right)^{\frac{1}{3}}.$$

Therefore, if $x^3 + px = q$ be identical with (21), then $B' = \frac{1}{27}p^3$, $A' = q$, and

$$x = \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

which is the well known CARDAN formula.

If in (17), $(x^3 + a^3)(x^3 + b^3)$ had been taken, the same result would have been obtained.

If in (17), $(x^3 + a^3)(x^3 - b^3)$ had been taken, then

$$(23) \quad A^3 + 3AB = A'$$

would result, which corresponds to $x^3 - px = q$, and

$$(24) \quad x = \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

Equations (22) and (24) are the same as $A = a + b$. And from the values of a and b , the values of $\frac{a}{2} \pm \frac{a}{2}\sqrt{-3}$, $\frac{b}{2} \pm \frac{b}{2}\sqrt{-3}$ can be obtained; also, from the various combinations of the six values which all enter into (17), its different quadratic factors might be constructed. If B' in that equation be supposed to arise from $B.B.B$, then A must correspond to such a combination of the values. Now in such case the combination is

$$\begin{aligned} &(-a)(-b); \left(\frac{a}{2} + \frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2} - \frac{b}{2}\sqrt{-3}\right); \\ &\left(\frac{a}{2} - \frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2} + \frac{b}{2}\sqrt{-3}\right). \end{aligned}$$

And from such a combination the corresponding values of A can be constructed; and these are the different values of A or x in (21) and (23), and correspond with the CARDAN formulas. By comparing the values of A as thus constructed with the forms as given in (5), we find, by reduction, that the roots of an equation of the form $x^3 + Bx + C = 0$, (25) are

$$(26) \quad x = (a+b), \quad x = -\frac{a+b}{2} + \frac{a-b}{2}\sqrt{-3}, \quad x = -\frac{a+b}{2} - \frac{a-b}{2}\sqrt{-3},$$

in which

$$a = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \frac{B^3}{27}}\right)^{\frac{1}{3}}, \quad b = \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \frac{B^3}{27}}\right)^{\frac{1}{3}}.$$

If the roots are all real, it is evident that $a - b$ must be imaginary. But this occurs when B is negative, and $\frac{B^3}{27} > \frac{C^2}{4}$, and the values of a and b are not then obtainable by a direct reduction, and the equation is then said to belong to the irreducible case. The method of

obtaining the values of a and b in such case, is by a development of their values. If

$$(27) \quad m = -\frac{4B^3 + 27C^2}{C^2},$$

the value of x in (25) can be put under the form

$$(28) \quad x = (-4C)^{\frac{1}{3}} \left(1 - \frac{1}{q} m^{\frac{1}{3}} \sqrt{-3}\right)^{\frac{1}{3}} + (-4C)^{\frac{1}{3}} \left(1 + \frac{1}{q} m^{\frac{1}{3}} \sqrt{-3}\right)^{\frac{1}{3}},$$

or by developing

$$(29) \quad x = (-4C)^{\frac{1}{3}} \left(1 + \frac{1}{3^{\frac{1}{3}}} m - \frac{10}{3^{\frac{2}{3}}} m^2 + \frac{154}{3^{\frac{5}{3}}} m^3 - \frac{935}{3^{\frac{8}{3}}} m^4 + \frac{11.13.17.23}{3^{\frac{11}{3}}} m^5 - \frac{8.13.17.23.29}{3^{\frac{14}{3}}} m^6 + \frac{17.20.23.29.38}{3^{\frac{17}{3}}} m^7 - \&c.\right);$$

or,

$$(30) \quad x = (-4C)^{\frac{1}{3}} \left(1 + \frac{m}{\log^{-1} 2.385606} - \frac{m^2}{\log^{-1} 4.247141} + \frac{m^3}{\log^{-1} 5.921154} - \frac{m^4}{\log^{-1} 7.523670} + \frac{m^5}{\log^{-1} 9.085424} - \frac{m^6}{\log^{-1} 10.622890} + \frac{m^7}{\log^{-1} 12.143496} - \&c.\right).$$

Equations (29) and (30) appear to give the value of the maximum root, and these series are rapidly converging, when m is a small number. If $m=0$, the two minor roots are equal, and the maximum root equals $(-4C)^{\frac{1}{3}}$. If both terms of the values of x in (28) be developed in the descending powers of the imaginary, the values of x become

$$(31) \quad x = -(-4C)^{\frac{1}{3}} \left(m^{-\frac{1}{3}} - 5m^{-\frac{4}{3}} + 66m^{-\frac{7}{3}} - 17.66m^{-\frac{10}{3}} + 17.23.55m^{-\frac{13}{3}} - 17.23.29.39m^{-\frac{16}{3}} + 17.23.24.29.35m^{-\frac{19}{3}} - \&c.\right).$$

This equation appears to give the minimum root, and will be converging when m is a large number. If one term of the value of x in (28) be dropped in the ascending powers of the imaginary, and the other in the descending, then the terms involving the imaginary cannot cancel each other, for the sum of the real terms in that case equals the half sum of (29) and (30), which is not a root of (25). Nor are we to suppose that the sum and difference of any number

of imaginary terms are equal to a real number. Therefore the expression for the value of x in this case is part real and part imaginary. But in the irreducible case now under consideration, all the roots are real; therefore this value of x is not a root of (25), but is still to be found among the combinations of the six roots or values which go to make up the quadratic factor of (17). If the values of x in (26) be combined, there results

$$(32) \quad x^3 - 3abx - a^3 - b^3 = 0,$$

from which, by comparing with $x^3 + Bx + C = 0$, we have

$$(33) \quad -3ab = B, \text{ or } -27a^3b^3 = B^3; \quad -a^3 - b^3 = C.$$

But it is evident, that, if $-\frac{a}{2} \pm \frac{a}{2}\sqrt{-3}$, $-\frac{b}{2} \pm \frac{b}{2}\sqrt{-3}$ were substituted in the first set of (33), $-a^3 - b^3 = C$ would be the same. The same would be the case if $-\frac{B}{2} \pm \frac{B}{2}\sqrt{-3}$ were substituted. Therefore the values of a and b derived therefrom involve these various cases, and the equations

$$(34) \quad \begin{aligned} x^3 + Bx + C = 0, \quad x^3 + \left(-\frac{B}{2} + \frac{B}{2}\sqrt{-3}\right)x + C = 0, \\ x^3 + \left(-\frac{B}{2} - \frac{B}{2}\sqrt{-3}\right)x + C = 0 \end{aligned}$$

are all equally solved by the CARDAN formulas.

It results from these developments, that an expression of the form $(m + n)^{\frac{1}{3}}$ does not give always the same value, when developed according to the ascending powers of n , as when developed according to the ascending powers of m .

If an equation of the fourth degree be examined with reference to its quadratic factors; as, for instance, if

$$(35) \quad (x^2 + Ax + B)(x^2 + Cx + D) = x^4 + ax^3 + bx^2 + cx + d = 0,$$

and the four roots are a', b', c', d' , it is plain that B, C, D , or A will have six values. For the values of B are $a'b', a'c', a'd', b'c', b'd', c'd'$.

Thence the value of any one of the coefficients of either of the quadratic factors, when obtained, should be expressed in the terms of an equation of the sixth degree. By equating, we get

$$(36) \quad A + C = a, \quad AC + B + D = b, \quad BC + AD = c, \quad BD = d.$$

By eliminating to find the value of B , we have, after reduction,

$$(37) \quad B^6 - bB^5 + (ac - d)B^4 + (2bd - c^2 - a^2d)B^3 \\ + (acd - d^2)B^2 - bd^2B + d^3 = 0.$$

Hence, assuming

$$(38) \quad x^6 + fx^5 + gx^4 + hx^3 + (gl^3 - 2l^3)x^2 + fl^3x + l = 0,$$

and equating the coefficients of (37) and (38), the values of a, b, c, d , the coefficients of (35), can easily be obtained. Hence the roots of an equation of the sixth degree of the form (38), as also its reciprocal, with their various modifications, are dependent upon the roots of the general bi-quadratic (35); for the roots of (38) are the roots of (35) multiplied together, two in a set.

This method of examining an equation by its factors, furnishes a general method of solution of equations; the only objection being, that, as we reason from the less degree, we are bound to accept the greater degree, in whatever form the equation may present itself. And if any equation be taken as the subject of examination, the question put is, What are the equations of the higher degrees, whose forms depend for solution upon the equation under examination? It may be said, that there are certain forms of equations of all degrees above the second, which may be evolved from equations of the second degree, and their solution thereby obtained. As, for instance, the equation of the fifth degree of the form

$$(39) \quad x^5 + Bx^3 + \frac{B}{5}x + C = 0$$

has for the value of x ,

$$(40) \quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}};$$

and its reciprocal,

$$(41) \quad x^5 + \frac{B^2}{56} x^4 + \frac{B}{C} x^3 + \frac{1}{C} = 0,$$

is therefore solved also.

Also, the form of the equation of the seventh degree is

$$(42) \quad x^7 + B x^5 + \frac{2}{7} B^2 x^3 + \frac{1}{49} B^3 x + C = 0,$$

its solution is

$$(43) \quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{7}\right)^7} \right)^{\frac{1}{7}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{7}\right)^7} \right)^{\frac{1}{7}};$$

its reciprocal is

$$(44) \quad x^7 + \frac{B^3}{C} x^6 + \frac{2}{7} B^2 x^4 + \frac{B}{49 C} x^2 + \frac{1}{C} = 0.$$

In all these cases, there are only two arbitrary quantities, B and C , from which to determine the coefficients; but by the introduction of the transformation $x' = y x$, the coefficients may be variously modified. The square root of an equation of the fifth degree of the general form will probably evolve an equation of the eighth degree; and if so, it will lack but one arbitrary quantity, with the assistance of the transformations, of showing the dependence of the general equation of the eighth degree upon one of the fifth.

If an equation of the third degree of the form

$$(45) \quad x^{15} - A' x^{10} + B' x^5 - C' = 0$$

be resolved into its five factors, by the method as above, we obtain

$$(46) \quad A^5 - 5 B A^3 + 5 C A^2 + 5 B^2 A - 5 B C = A';$$

$$(47) \quad B^5 - 5 A C B^3 + 5 C^2 B^2 + 5 A^2 C^2 B - 5 A C^3 = B';$$

$$(48) \quad C^5 = C',$$

from which it will require some industry to eliminate A, B, C .

There is but one other matter to be named; and that is as to the question, How are the imaginary roots to be interpreted? In the quadratic equation, when the roots become imaginary, the conditions are said to be impossible, and they are used to detect such

impossible conditions. But in equations of a degree above the second, they may be part real and part imaginary; and in such case, how is the test to be used? For in such case, the imaginary roots are as truly roots of the equation as the real roots. There is but little doubt that they admit of geometrical interpretation. Otherwise, equations of the higher orders, instead of being of extraordinary power, from their extreme flexibility and great capacity, may be said to be labyrinths filled with nests of ghostly quantities.

Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the May number of the MONTHLY: —

C. M. WOODWARD, Junior Class, Harvard College, answered all the questions. (BENJAMIN PEIRCE, Prof.)

HORACE OTIS, Adams Centre, N. Y., answered I. and IV.

P. BARTON, Amsterdam, N. Y., answered all but III.

ROLAND THOMPSON, Junior Class, Jefferson College, Cannonsburg, Penn., answered I. and IV. (JOHN FRASER, Prof.)

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered all the questions.

W. F. OSBORNE, Sophomore Class, Wesleyan University, Middletown, Conn., answered all the questions. (J. M. VAN VLECK, Prof.)

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions.

GUSTAVUS FRANKENSTEIN, Springfield, Ohio, answered all the questions.

GEORGE A. OSBORNE, Jr., Lawrence Scientific School, answered all the questions. (H. L. EUSTIS, Prof.)

GEORGE W. JONES, Jr., Senior Class, Yale College, answered all the questions. (H. A. NEWTON, Prof.)

W. MURRAY STERLING, Student, Baltimore, Md., answered questions I., II. and IV. (TIMOTHY CRIMMIN, Teacher.)

A STUDENT, High School, Baltimore, Md., answered questions I., II. and IV. (JAMES MCINTIRE, Prof.)

It gives us pleasure to add the following names to our list of coöperators and contributors: EDWIN HAAS, Burlington, N. J.; R. C. MATTHEWSON, San Francisco, Cal.; W. C.

DENNIS, Key West, Florida; AUGUST SONNTAG, Acting Assistant at the Dudley Observatory, Albany, N. Y.; WILLIAM J. LEWIS, San Francisco, Cal. . . . To the following Card, which we are permitted to lay before the readers and friends of the *MATHEMATICAL MONTHLY*, we have nothing to add, except that it amply repays us for the care and labor we have already devoted to the work. All we ask, is such a support as shall enable us to publish promptly all we receive worthy of being put in more permanent form:—

“Boston, May 25th, 1859.

“The undersigned, having watched with great interest the establishment of a new *MATHEMATICAL JOURNAL* at Cambridge, under the editorship of Mr. J. D. RUNKLE, are desirous of calling the attention of the patrons of sound learning to this work. The ‘*MATHEMATICAL MONTHLY*’ commenced its existence in October, 1858, and has been conducted on a plan that cannot fail to make it an instrument of great good. It is addressed to students as well as to professors; and has doubtless already given a new impulse to mathematical studies, wherever it has been introduced. A work of this kind should not be suffered simply to live. It has now about eleven hundred subscribers; and is not, perhaps, likely to be altogether discontinued, while its present support remains. But its friends should not be satisfied with this. A liberal subscription should insure to it a vigorous, energetic, long-continued life; and should enable it not only to preserve its present excellent form, and the stimulus of its prizes,—now amounting to about three hundred dollars,—but to make new improvements; to increase its size, without increasing its terms; to secure the best matter, by paying contributors, if necessary; and to compensate its editor, in part at least, for the time he bestows upon it.

“This appeal is made from no suggestion of the editor, but from the conviction that the work deserves a wide patronage, and must secure it, in order to be *permanently* successful. The circle of strictly mathematical readers in this country is yet small. To enlarge it, nothing can be more surely relied on than a well-conducted Journal; but until this is done, and the work is thus made to support itself, the aid of *all* true friends of Science must be invoked.

“JAMES WALKER.
JARED SPARKS.
BENJAMIN PEIRCE.
JOSEPH LOVERING.
G. P. BOND.
JOSIAH QUINCY.
EDWARD EVERETT.
J. INGERSOLL BOWDITCH.”

We hope that Prof. WILLIAM RUTHERFORD, of the Royal Military Academy, Woolwich, England, will pardon us for giving, in connection with the above, an extract from his polite note of May 4th, 1859:—

“I have also got all the numbers of your new *Monthly Mathematical* publication, with which I am very much pleased. It will be a useful work, and I regret we have not any similar publication in Britain.”

Prof. GIBBES has sent us the following errata: On page 338, last line, for > 0.250 , read < 0.250 ; page 339, line —5, in the value of y , put exponent $\frac{1}{2}$ outside the parenthesis. A few unimportant verbal errors we omit, as they will give the reader no difficulty. On page 350, for 63 cents, read 36.

SCIENTIFIC BOOKS,

IMPORTED AND FOR SALE BY

SEVER AND FRANCIS,...CAMBRIDGE,

SUCCESSORS TO JOHN BARTLETT.

* * Orders for the Importation of English, French, and German Books will be promptly executed,
and at moderate prices.

BOOKS FOR PUBLIC LIBRARIES IMPORTED FREE OF DUTY.

- ADHEMAR. Le Traité des Ponts biais. 8vo, et atlas in fol. \$8.00.
- ANNALES DE L'OBSERVATOIRE IMPERIAL DE PARIS. Par Le Verrier. 3 vols. in 4to, hf. cf. \$24.00.
- ARMSTRONG. Steam Boilers. 30c.
- AUDÉ. Poussée des Terres. 8vo, hf. cf. \$1.87.
- BAKER. Treatise on Statics and Dynamics. 30c.
- . Treatise on Land Surveying and Engineering. 30c.
- BARLOW. On Materials and Construction. 8vo. \$4.80.
- BIOT. Refractions Atmospheriques. 4to. \$1.25.
- . Traité des Equations différentielles. Hf. mor. \$2.00.
- BLACK. Iron Highways from London to Edinburgh, &c. \$1.00.
- BLAND. Algebraical Problems. 8vo. (Second-hand.) \$1.50.
- . Key to do. 8vo. " " \$1.75.
- . Philosophical Problems. 8vo. " " \$1.50.
- BOOTH. Theory of Elliptic Integrals. 8vo. \$2.25.
- BORDEN. System of Railway Formulæ. 8vo. \$2.25.
- BOURCHARLAT. Elements de Calcul differential, et du Calcul integral. 8vo. \$2.25.
- BOURDON. Application de l'Algèbre a la Géométrie. 8vo, hf. cf. \$3.00.
- BOURNE. Treatise on the Steam Engine. 4to. \$6.00.
- BOW. Treatise on Bracing. 8vo. \$1.12.
- BRANDE. Dictionary of Science, Literature, and Art. 8vo. \$7.50.
- BRETON. Traité du Nivellement. 8vo. \$2.50.
- BUCK. Oblique Arches of Bridges. 4to. \$3.75.
- BURGOYNE. Blasting and Quarrying Stone. 30c.
- BURNELL. Treatise on Limes and Cements. 30c.
- BURY. Rudimentary Treatise on Architecture. 45c.
- CALLET. Tables de Logarithmes. 8vo, hf. cf. \$5.00.
- CARMICHAEL. Treatise on Calculus of Operations. 8vo. \$2.75.
- CHARLES. Traité de Géométrie Supérieure. 8vo, hf. cf. \$6.00.
- CHÉUVENET. Trigonometry. 8vo.
- CHOQUET. Traité élémentaire d'Algèbre. 8vo, hf. cf. \$2.00.
- CIVIL ENGINEER AND ARCHITECT'S JOURNAL. Monthly. 4to. London. \$7.50 per annum, post-paid.
- CLARK. Britannia and Conway Tubular Bridges. 2 vols. 8vo, et atlas folio. \$28.50.
- CLEGG. Architecture of Machinery. 4to. \$2.00.
- CRELLE. Journal für die reine und angewandte Mathematik. \$4.00 per annum.
- COURNOT. Traité élémentaire de la theorie des fonctions, et du Calcul infinitesimal. 2 vols. 8vo. \$4.00.
- COURTENAY. Calculus. 8vo.
- CRESWELL. Treatise on Spherics. 8vo. (Second-hand.) \$1.00.
- CRESY. On Bridge Building, and Equilibrium of Vaults and Arches. Folio. \$10.00.
- . Cyclopædia of Engineering. 8vo. \$15.00.
- D'AUBUISSON. Treatise on Hydraulics. Translated by Joseph Bennett. 8vo.
- DE LA RIVE. Traité d'Electricité theorique et appliquée. 3 vols. 8vo, hf. mor. \$9.00.
- . Treatise on Electricity, in Theory and Practice. 3 vols. 8vo. \$22.00.
- DELISLE. Géométrie Analytique. 8vo, hf. cf. \$3.00.
- DE MORGAN. Differential and Integral Calculus. 8vo. \$2.00.
- DEMPSEY. Treatise on Drainage and Sewerage of Towns and Buildings. 45c.
- . Treatise on Drainage and Sewerage of Districts and Lands. 30c.
- DOBSON. On Foundations and Concrete Works. 30c.
- DOBSON. On Masonry and Stonecutting. 60c.
- . On Bricks and Tiles. 30c.
- DUHAMEL. Cours d'Analyse Calcul infinitésimal. 2 vols. 8vo, pp. 300; hf. mor. \$5.00.
- . Cours de Mécanique. 2 vols. 8vo, pp. 300, hf. mor. \$4.50.
- DUPIN. Application de Géométrie et de Mécanique à la Marine aux Ponts et Chaussées. 4to, hf. mor. \$4.50.
- . Developpements de Géométrie. 4to, hf. mor. \$4.50.
- . Géométrie et Mécanique, appliquée aux arts. 8vo, hf. cf. \$2.50.
- EARNSHAW. Dynamics. 8vo. (Second-hand.) \$1.50.
- EMY. Traité de la Charpenterie. 2 vols. in 4to, et atlas in fol. hf. mor. \$25.00.
- FAIRBAIRN. On the Application of Cast and Wrought Iron to Building Purposes. 8vo. \$3.50.
- . Useful Information for Engineers. 8vo. \$3.25.
- FERGUSON. Handbook of Architecture. 2 vols. 8vo, hf. cf. \$12.
- FISCHER. Logarithmic Tables of Seven Places. Translated from Bremiker's Vega. 8vo, hf. mor. \$2.50.
- FRANCOEUR. Cours complet de Mathématiques pures. 2 vols. 8vo. \$5.00.
- GANOT. Traité élémentaire de Physique. 12mo. \$1.75.
- GARBETT. Principles of Design in Architecture. 60c.
- GAUSS. Méthode des moindres carrés. 8vo, hf. cf. \$1.62.
- . Theoria Motus. Translated by C. H. Davis. 4to. \$5.00.
- GATFFIERS. Manual des Ponts et Chaussées. 2 vols. 18mo, hf. cf. \$3.00.
- GILLESPIE. Land Surveying. 8vo.
- . Roads. 8vo.
- GODFRAY. The Lunar Theory. 8vo. \$1.62.
- GRANT. Plane Astronomy. 8vo. \$1.75.
- GREGORY. Differential and Integral Calculus, by Walton. 8vo. \$5.50.
- HADDON. Differential Calculus. 30c.

SCIENTIFIC BOOKS.

- HAMILTON. Lectures on Quaternions. 8vo. \$6.30.
HANN. Integral Calculus. 30c.
HANN AND GENNER. On the Steam Engine. 8vo. \$2.75.
HART. On Oblique Arches. 4to. \$2.40.
HASKOLL. Railway Construction, from the setting out of the centre line to the completion of the work. 2 vols. Roy. 8vo. Plates. \$12.50.
HEATHER. Descriptive Geometry. 30c.
HEMMING. Differential and Integral Calculus. 8vo. \$2.75.
HENCK. Fieldbook for Engineers. 18mo.
HUGHES. Treatise on Gas Works. 90c.
HUTTON. Mathematics. 8vo. \$4.00.
———. Recreations. 8vo. \$4.00.
HUTTON. Mathematical Tables. 8vo. \$3.75.
HYMER. Plane and Spherical Trigonometry. \$2.50.
———. Algebraic Equations. 8vo. \$3.00.
———. Analytical Geometry. \$3.00.
———. Differential Equations, and Calculus of Finite Differences. 8vo. \$3.60.
———. Integral Calculus. 8vo. \$3.00.
———. Geometry of Three Dimensions. 8vo. Cloth. \$3.00.
———. Conic Sections. 8vo. (Second-hand.) \$1.25.
JAMIESON. Solutions of the Senate House Rider. 8vo. \$2.25.
JELLETT. Calculus of Variations. 8vo. \$4.50.
KÜHLER. Logarithmisch Trigonometrisches Handbuch. 8vo. \$2.12.
LACROIX. Traité du Calcul Differential et du Calcul Integral. 3 vols. in 4to, avec 18 pl. Mor. fine copy. \$35.00.
LAGRANGE. Mécanique Analytique. 2 vols. 4to, hf. mor. \$13.00.
———. Théorie des Fonctions Analytiques. 4to, hf. mor. \$6.00.
———. Traité des Equations Numeriques de tous les degrés. 4to, hf. mor. \$5.00.
LALANDE. Tables de Logarithmes. 18mo. 60c.
LAPLACE. Systeme du Monde. 4to, hf. mor. \$5.50.
LAW. Civil Engineering. \$1.37.
———. Constructing and Repairing Roads. 30c.
LEGENDRE. Traité des Fonctions Elliptiques. 3 vols. 4to, hf. mor. \$20.00.
LEROY. Traité de Stéréotomie comprenant les Applications de la Géométrie Descriptive a la Theorie des Ombres, la Perspective Lineaire, la Gnomonique, la Coupe des Pierres et la Charpente. In 4to, avec atlas de 74 pl. in fol. Paper, \$9.00. Hf. mor. \$12.00.
———. Traité de Géométrie Descriptive. In 4to, avec atlas de 71 pl. Paper, \$3.00. Hf. mor. \$5.00, \$6.00.
———. Analyse Appliquée a la Géométrie des trois dimensions. 8vo, hf. cf. \$2.25.
LIOUVILLE. Journal de Mathematiques pures et appliquées. 1836-1857. 22 vols. 4to, hf. cf. \$150.00.
LOUDON. Cyclopædia of Architecture. 8vo. \$10.
MAHAN. Civil Engineering. 8vo.
MAIN AND BROWN. Indicator and Dynamometer. \$1.40.
MONGE. Application de l'Analyse a la Geometrie. 4to, hf. mor. \$10.50.
———. Traité élémentaire de Statique. 8vo, hf. mor. \$1.75.
MORIN. Aide-memoire de Mécanique Pratique. 8vo. \$2.00.
MOSELY. Mechanics of Engineering. 8vo.
MULLER. Physics and Meteorology. 8vo. \$4.00.
NAVIER. (Ketten) Hängbrücken. 4to. \$5.00.
NEVILLE. Hydraulic Formulæ. 8vo. \$2.75.
NICOL. Cyclopædia of the Physical Sciences. 8vo. \$3.75.
OLIVIER. Theorie géométrique des Engrenages. 4to, hf. mor. \$3.75.
———. Developpements de Géométrie Descriptive. 2 vols. 4to, hf. mor. \$9.00.
———. Complément de Géométrie Descriptive. 2 vols. 4to, hf. mor. \$9.00.
———. Mémoires de Géométrie Descriptive, Theorique, et Appliquée. 2 vols. 4to, hf. mor. \$9.00.
———. Application de la Géométrie Descriptive. 2 vols. 4to, hf. mor. \$10.00.
PARKINSON. Elementary Mechanics. Post 8vo. \$2.87.
PEIRCE. Analytic Mechanics. 4to, cloth. \$7.50.
———, J. M. Analytic Geometry. 8vo. \$1.50.
PHEAR. Elementary Hydrostatics. Post 8vo. \$1.62.
POINSON. Eléments de Statique. 8vo, hf. cf. \$2.50.
POISSON. Mécanique. 2 vols. 8vo. \$6.00.
PONCELET ET LESBROS. Experiences Hydrauliques. 4to, hf. mor. \$4.00.
PONTECOULANT. Systeme du Monde. 4 vols. 8vo, hf. cf. \$16.00.
POUILLET. Elements de Physique Experimentale. 3 vols. 8vo, hf. cf. \$7.50.
PRALY. Construction des Voutes biaises. 8vo, hf. cf. \$1.87.
PRICE. Differential and Integral Calculus. 3 vols. 8vo. \$14.50.
QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS. Sylvester, Feriers & Co. \$6.00 per annum. Post paid.
———. 1855-8. 8 Nos. \$12.00.
SERRET. Cours d'Algèbre supérieure. 8vo, hf. cf. \$3.50.
SEWELL. Treatise on Steam and Locomotives. 30c.
SIMMS. On the Use of Instruments. 8vo.
SIMMS. On Levelling. 8vo.
SMITH. Linear Drawing. 8vo.
———. Topographical Drawing. 8vo.
SOLUTIONS OF THE CAMBRIDGE PROBLEMS. 1848 to 1851. Ferrers and Jackson. 8vo. \$4.50.
———. 1854. Walton and Mackenzie. 8vo. \$3.25.
———. 1857. Cam-pion and Walton. 8vo. \$2.50.
———. Senate House Riders. Jamieson. 8vo. \$2.25.
SPOTTISWOODE. Elementary Theorems relating to Determinants. 4to. \$1.50.
STEVENSON. Treatise on Lighthouses. 90c.
SWINDELL. Well Digging and Boring. 30c.
TATE. Exercises on Mechanics. With Key. 2 vols. \$1.62.
———. Mechanical Philosophy. 8vo. \$3.25.
———. Materials. 8vo. \$1.75.
TATE AND STEEL. Dynamics. Post 8vo. \$3.00.
TERQUEM. Nouvelles Annales de Mathématiques. Monthly. \$4.00 per annum.
———. Nouvelles Annales de Mathématiques, 1856-7. 2 vols. Hf. calf. \$6.50.
TODHUNTER. Analytical Statics. Post 8vo. \$3.00.
———. Differential and Integral Calculus. 2 vols. Post 8vo. \$6.00.
TREDGOLD. Strength of Materials. Edited by Hodgkinson. 2 vols. 8vo. \$7.50.
TRESCA. Géométrie Descriptive. 8vo, hf. mor. \$3.00.

SCIENTIFIC BOOKS.

- URE. Dictionary of Arts, Manufactures, and Mines. 2 vols. 8vo. \$5.00.
- VALLEE. Traité de la Science du Desin. Contenant la Theorie Générale des Ombres, la Perspective Lineaire, la Theorie Générale des Images d'optique et le Perspective Aérienne appliquée au Lavis, pour faire suite a la Géométrie Descriptive. In 4to, et atlas de 56 pl. Hf. mor. \$5.50.
- VERHULST. Traite élémentaire des Fonctions Elliptiques. 8vo, hf. cf. \$2.75.
- VIEILLE. Cours complémentaire d'Analyse et de Mécanique. 8vo, hf. cf. \$2.50.
- VICAT. On Cements. Translated by Capt. Smith.
- WALTON. Problems illustrative of Plane Coördinate Geometry. 8vo. \$4.80.
- . Problems, Mechanical Collection of. 8vo. \$5.50.
- . Problems in Hydrostatics and Hydrodynamics. 8vo. \$3.00.
- . On the Differential Calculus. 8vo. \$3.25.
- WALTON AND MACKENZIE. Solutions of the Cambridge Problems. 8vo. 1854, \$3.25. do. 1857, \$2.50.
- . Problems in Elementary Mechanics. 8vo, cloth. 1858. \$3.00.
- WARR. Dynamics, Construction of Machinery. 8vo. \$2.87.
- WHEWELL. Analytical Statics. 8vo. \$2.25.
- WIESBACH. Mechanics. 2 vols. 8vo.
- WILLIAMS. Practical Geodesy. Post 8vo. \$2.50.
- WILME. Handbook for Mapping, Engineering, &c. 4to, plain, \$7.00. Colored, \$12.00.
- WILSON'S DYNAMICS. 8vo. \$2.87.
- WOLFER. Tabulæ Reductionum Observationum Astronomica-rum. 8vo. \$3.50.
- WOOD. Practical Treatise on Railroads. 8vo, hf. mor. \$4.00.
- WOOLHOUSE. Weights and Measures of all Nations. 30c.
- . Differential Calculus. 30c.
- WORTHEN. Appleton's Handbook of Drawing. 8vo.
- YVON, VILLARCEAU. Etablissement des Arches de pont. 4to. hf. mor. \$5.00.

* * * Early copies of all the new mathematical books published in England and France will be received as soon as published.

LIST OF BOOKS RECEIVED,

FOR SALE,

SINCE JANUARY, 1859.

- ARAGO. Astronomie. 4 vols. 8vo, hf. mor. \$12.00.
- . Notices Biographiques. 3 vols. 8vo, hf. mor. \$9.00.
- . Notices Scientifiques. 4 vols. 8vo, hf. mor. \$12.00.
- . Memoires Scientifiques. vol. 1. 8vo, hf. mor. \$3.00.
- . Voyages. 8vo. hf. mor. \$3.00.
- BONNET. Leçons de Mécanique Élémentaire a l'usage des Candidats a l'Ecole Polytechnique. 8vo. \$1.25.
- BRIOSCHE. Théorie des Determinants. 8vo. \$1.25.
- BRITISH NAUTICAL ALMANAC, 1861. 8vo. \$1.00.
- , 1862. 8vo. \$1.00.
- CARMICHAEL. Calculus of Operations. 8vo. \$2.75.
- COOMBE. Solutions of the Cambridge Problems, 1840-1841. 8vo. \$2.25.
- FRANCOEUR. Éléments de Statique. 8vo, hf. cf. \$2.00.
- FROST. Mathematical Questions of the Senate House Examination Papers, 1838 to 1849. 8vo. \$3.00.
- GANOT. Physique a l'usage des Gens du Monde, 308 magnifiques vignettes. hf. cf. \$2.00.
- JAMIN. Cours de Physique, de l'Ecole Polytechnique. 8vo. \$3.00.
- LATEAM. Construction of Wrought Iron Bridges. Embracing the Practical Application of the Principles of Mechanics to Wrought Iron Girder Work. With numerous detail plates. 8vo. \$4.75.
- MAHISTRE. Cours de mécanique appliquée. 8vo. \$2.00.
- MATHEMATICAL PROBLEMS and Examples of the Senate House Examination Papers, 1821-1836, with an appendix containing the Senate House Questions for 1837. 8vo. \$3.00.
- MONTFERRIER. Encyclopédie Mathématique ou Exposition complète de Toutes les Branches des Mathématiques d'après les principes de la Philosophie des Mathématiques de Hoëné Wronski. 3 vols. 8vo. \$6.00. To be continued in Monthly Livraisons.
- OLIVIER. Cours de Geométrie Descriptive. 2 vols. 4to. hf. mor. \$10.00.
- QUARTERLY JOURNAL of Pure and Applied Mathematics. November, 1858. No. 1. Vol. 3. \$1.50.
- RANKINE. Manual of Applied Mechanics, 1858. \$3.00.
- REGNAULT. Manuel des Aspirants au grade d'Ingénieur des Ponts et Chaussées. Partie théorique et Partie pratique. 4 vols. 8vo. hf. cf. \$10.00.
- SALMON. Conic Sections. \$3.50.
- . Higher Plane Curves. \$3.50.
- SERRET. Éléments d'Arithmétique. 8vo. hf. cf. \$1.50.
- SONNET. Premiers Éléments de Mécanique Appliquée. 12mo. hf. cf. \$1.75.
- TERQUEM. Nouvelles Annales de Mathématiques. Tomes VIII. a XVII. Corresponding to the years 1849 to 1859. 10 vols. hf. cf.
- . Bulletin de Bibliographie, d'Histoire et de Biographie Mathématiques. 8vo. hf. cf. \$3.00.

THE NATIONAL SERIES OF MATHEMATICS.

BY CHARLES DAVIES, LL. D.,

PROFESSOR OF MATHEMATICS IN COLUMBIA COLLEGE, N. Y.

PUBLISHED BY A. S. BARNES & CO., 51 & 53 JOHN STREET, NEW YORK.

Elementary Course.

	Retail price.
DAVIES' PRIMARY ARITHMETIC & TABLE BOOK.	\$0 15
DAVIES' FIRST LESSONS IN ARITHMETIC.	0 20
DAVIES' INTELLECTUAL ARITHMETIC.	0 25
DAVIES' NEW SCHOOL ARITHMETIC.	0 45
KEY TO DAVIES' NEW SCHOOL ARITHMETIC.	0 45
DAVIES' NEW UNIVERSITY ARITHMETIC.	0 75
KEY TO DAVIES' NEW UNIVERSITY ARITHMETIC.	0 50
DAVIES' GRAMMAR OF ARITHMETIC.	0 30
DAVIES' ELEMENTARY ALGEBRA.	0 75
KEY TO DAVIES' ELEMENTARY ALGEBRA.	0 50
DAVIES' ELEMENTARY GEOMETRY & TRIGONOM.	1 00
DAVIES' PRACTICAL MATHEMATICS.	1 00

Advanced Course.

	Retail price.
DAVIES' UNIVERSITY ALGEBRA.	\$1 25
KEY TO DAVIES' UNIVERSITY ALGEBRA.	1 00
DAVIES' BOURDON'S ALGEBRA.	1 50
KEY TO DAVIES' BOURDON'S ALGEBRA.	1 50
DAVIES' LEGENDRE'S GEOMETRY.	1 50
DAVIES' ELEMENTS OF SURVEYING.	1 50
DAVIES' ANALYTICAL GEOMETRY.	1 25
DAVIES' DIFFERENTIAL & INTEGRAL CALCULUS.	1 25
DAVIES' DESCRIPTIVE GEOMETRY.	2 00
DAVIES' SHADES, SHADOWS, AND PERSPECTIVE.	2 50
DAVIES' LOGIC OF MATHEMATICS.	1 25
DAVIES' & PECK'S MATHEMATICAL DICTIONARY.	2 50

A. S. BARNES & Co. have the pleasure of announcing, that they have just issued from their press AN ENTIRELY NEW WORK ON ALGEBRA, by PROFESSOR DAVIES, entitled

DAVIES' UNIVERSITY ALGEBRA.

This work is designed to occupy an intermediate place between his Elementary Algebra and Bourdon. It teaches the Science and Art of Algebra by a logical arrangement and classification of the principles in their natural order, and by illustrating their application

in an extended series of carefully arranged and graded examples. It is well adapted for use in High Schools, Academies, and Colleges; the work being so divided and arranged that it may be studied in parts, or as a whole, forming a full and complete course.

NATURAL PHILOSOPHY.

PARKER'S JUVENILE PHILOSOPHY.	Price \$0 25
PARKER'S FIRST LESSONS IN PHILOSOPHY.	0 37½
PARKER'S COMPENDIUM OF SCHOOL PHILOSOPHY.	1 00

The present edition of Parker's School Philosophy has been corrected, enlarged, and improved, and contains all the late discoveries and improvements in the science up to the present time.

It contains engravings of the Boston School set of apparatus; a description of the instruments, and an account of many experiments which can be performed by means of the apparatus, — and it is peculiarly adapted to the convenience of study and recitation. The work is immensely popular, and in very extensive use, more so than any other work of the kind. *It has been recommended by the Superintendents of Public Instruction of six States, and is the Standard Text-Book in all the principal cities of the United States, and throughout Canada West.*

NORTON'S FIRST BOOK OF PHILOSOPHY AND ASTRONOMY.

By WILLIAM A. NORTON, M. A., Professor of Civil Engineering in Yale College. Arranged upon the catechetical plan, and copiously illustrated. Designed for Young Pupils commencing the study of the science.

THE FIRST BOOK OF SCIENCE — TWO PARTS IN ONE. \$1 00

PART I. NATURAL PHILOSOPHY AND ASTRONOMY. PART II. CHEMISTRY AND ALLIED SCIENCES. By W. A. NORTON and J. A. PORTER, Professors in Yale College.

This volume treats of the elements of Natural Science, and is designed to meet the wants of young persons who do not intend to pursue a complete course of academical study. It is designed for Public and Private Schools, and will be found admirably adapted to private study, and home instruction in familiar science.

BARTLETT'S COLLEGE PHILOSOPHY.

BARTLETT'S SYNTHETIC MECHANICS.	\$3 00
BARTLETT'S ANALYTIC MECHANICS.	4 00
BARTLETT'S OPTICS AND ACOUSTICS.	2 00
BARTLETT'S SPHERICAL ASTRONOMY.	3 00

The above are the Text-books in the U. S. Military Academy at West Point.

PORTER'S SCHOOL CHEMISTRY.

First Book of Chemistry, and Allied Sciences, including an Outline of Agricultural Chemistry. By PROF. JOHN A. PORTER. Price 50 cts.

Principles of Chemistry, embracing the most recent discoveries in the Science, and the Outlines of its application to Agriculture and the Arts — illustrated by numerous experiments newly adapted to the simplest apparatus. By JOHN A. PORTER, A. M., M. D., Professor of Agricultural and Organic Chemistry in Yale College. Price \$1.

These works have been prepared expressly for Public and Union Schools, Academies, and Seminaries, where an extensive course of study on this subject and expensive apparatus were not desired, or could not be afforded. A fair, practical knowledge of Chemistry is exceedingly desirable, and almost a necessity at the present day, but it has been taught in very few Public or Union Schools, owing entirely to the want of suitable text-books adapted to simple apparatus, or such as could be readily obtained. It is confidently believed that these works supply this great want, and will be found in every respect just what is required. Boxes containing all the apparatus and materials necessary to perform all the experiments described in these books, can be obtained for \$8.00, by addressing A. S. BARNES & Co., New York.

IN PRESS.

Prof. PECK of Columbia College is preparing an Elementary Work on MECHANICS.

g
-
-
y
,
le

s,
b-
A
a
on
le
ed
et
oc-
bb-

NATIONAL SERIES OF STANDARD SCHOOL-BOOKS,

Published by A. S. BARNES & BURR, New York.

The attention of the Friends of Education is invited to the "NATIONAL SERIES OF STANDARD SCHOOL AND LIBRARY BOOKS," designed as Class-Books, for the use of Schools, Academies, Colleges, Families, and Libraries.

The Publishers would express their grateful acknowledgments to numerous Educators throughout the United States, for their patronage and kind expressions of appreciation of the merits of their Publications. It is their intention to use all possible endeavors to sustain the present reputation of these Works, by issuing the most approved editions within the range of School and Academic Instruction. The best talent that could be procured has been employed in the preparation of these Works; and the high standing they have already attained, as Class-Books for the Institutions of our country, is gratifying evidence of their intrinsic merit. The LIBRARY BOOKS attached to this Series will be found worthy of the high encomiums that have been given of them.

SCIENCE OF THE ENGLISH LANGUAGE.

SPELLING AND READING.

Pago's Normal Chart of Elementary Sounds,	12mo, half-bd.	\$3 50
Wright's Analytical Orthography,	" "	25
Smith's Juvenile Definer,	" "	30
Smith's Grammar School Speller,	" "	38
Smith's Definer's Manual,	" "	60

Parker and Watson's New Series.

National School Primer; or Primary Word-Builder,	18mo.	15
National Elementary Speller,	18mo, half-bd.	15
National Pronouncing Speller,	12mo, half-bd.	25
National First Reader; or, Word-Builder,	18mo, half-bd.	25
National Second Reader,	" "	38
National Third Reader,	" "	50
National Fourth Reader,	12mo, muslin.	75
National Fifth Reader,	" "	1 00
Parker's Series of School Readers,	5 Nos.	1 00
High School Literature,	" "	1 00
North Carolina Reader,	3 Nos.	1 00
Spanish, German, and French Readers. Illustrated.	Each	13

NATIONAL ELOCUTIONARY SERIES.

Parker and Zachos' Reading and Elocution,	12mo, half-bd.	40
Northend's Little Orator,	18mo.	30
Northend's National Orator,	12mo.	75
Northend's Youth's Dialogues,	12mo.	75
National University Orator,	8vo.	1 00

ENGLISH GRAMMAR.

Clark's Grammatical Chart,	mounted.	2 50
Clark's Analysis of the English Language,	half-bd.	40
Clark's First Lessons in English Grammar,	" "	30
Clark's New English Grammar,	revised edition.	60
Clark's Key to English Grammar,	In press.	
Welch's Analysis of the English Sentence,	12mo.	75

RHETORIC, LOGIC, &c.

Day's Elements of the Art of Rhetoric,	12mo.	75
Boyd's Kames' Elements of Criticism,	8vo.	1 25
Boyd's Elements of Composition,	" "	75
Mahan's System of Logic, for Colleges,	8vo.	1 50
Mahan's Intellectual Philosophy,	12mo.	1 00
Watts on the Mind: with Questions,	18mo	34
Willard's Morals for the Young,	16mo.	60

The English Poets, with Boyd's Notes.

Boyd's Thompson's Seasons,	School edition, half-bd., 12mo.	75
Boyd's Milton's Paradise Lost,	" "	75
Boyd's Young's Night Thoughts,	" "	75
Boyd's Cowper's Task, &c.	" "	75
Boyd's Pollok's Course of Time,	" "	75

National Geographical Series.

Monteith's First Lessons in Geography,	16mo.	No. 1.	25
Monteith's Introduction to Manual of Geography,	" "	No. 2.	40
Monteith's Manual of Geography. (Revised Edition),	" "	No. 3.	60
McNally's Complete School Geography,	4to.	No. 4.	1 00

History and Mythology.

Monteith's Youth's History of the United States,		50
Willard's School History of the United States. New Edition,		75
Willard's History of the United States.	8vo.	1 50
Willard's Historia de los Estados Unidos (Spanish language.)	2mo.	2 00
Willard's Universal History in Perspective,	8vo.	1 50
Dwight's Grecian and Roman Mythology,	12mo.	75
Ricordi's Roman History; with Questions,		1 25
Gould's Alison's History of Europe,		1 25
Sheldon's History of Michigan,		1 50

Church's Elements of Calculus,	1 vol. 8vo.	2 00
Church's Analytical Geometry,	" "	2 00
Courtenay's Differential and Integral Calculus,	" "	2 50
Hackley's Trigonometry,	" "	2 50
Reuck's Practical Examples in Arithmetic,	" "	60
Smith and Martin's Book-Keeping,	" "	75

Natural Philosophy and Chemistry.

Norton's First Book of Natural Philosophy and Astronomy,		60
McIntyre on the Use of the Globes,	12mo.	1 00
Peck's Elements of Mechanics,		1 50
Lardner on the Steam Engine,	8vo.	1 50
Gillespie on Roads and Railroads,	1 vol. 8vo.	1 50
Bartlett's Synthetical Mechanics,	" "	3 00
Bartlett's Treatise on Optics, &c.,	" "	2 00
Bartlett's Analytical Mechanics,	" "	4 00
Bartlett's Treatise on Astronomy. New edition.	" "	4 00
Darby's Southern Botany, for Southern States,	1 vol.	1 75

Brook's Latin and Greek Classics.

Brook's First Latin Lessons,	12mo.	62
Brook's Ovid Metamorphoses,	Illustrated. 8vo, sheep.	2 50
Brook's Collectanea Evangelica,	18mo.	62
Brook's First Greek Lessons,	18mo.	62
Brook's Historia Sacra,	18mo.	62
Brook's Caesar's Commentaries,	Illustrated. 12mo.	1 25

School Teacher's Library.

Northend's Teacher and Parent,	12mo, muslin.	1 25
Page's Theory and Practice of Teaching,	" "	1 25
Mansfield on American Education,	" "	1 25
DeTocqueville's American Institutions,	" "	1 25
Dwight on Christian Education,	" "	1 25
Mayhew on Universal Education,	" "	1 25
Root on School Amusements,	" "	1 25
Bates' Institute Lectures,	" "	1 25
Brook's School Teachers' Register,	" "	60

PROFESSOR BARTLETT'S COLLEGE TEXT-BOOKS.

NATURAL PHILOSOPHY.

"The Scientific Works of Professor BARTLETT, namely, ANALYTICAL MECHANICS, ACOUSTICS, OPTICS and ASTRONOMY, have been taught by the undersigned in the New York Free Academy, about four years. Their use has been attended with great success, both in communicating a higher tone of scholarship, and a deep interest in mathematical and philosophical studies.

"JOHN A. NICHOLS, Professor of Mixed Mathematics."

"The marked success that has attended the use of Prof. BARTLETT's Treatises on the several branches of Mixed Mathematics, in this Institution, justify the undersigned in recommending them as works possessing great excellence. The undersigned also concurs most cordially in the statement made by Prof. NICHOLS.

"New York Free Academy, June 24, 1859."

"HORACE WEBSTER, President Faculty of Free Academy."

PROFESSOR PECK'S WORK ON MECHANICS.

ELEMENTS OF MECHANICS, Designed to occupy an intermediate place between BARTLETT's Higher Course of Natural Philosophy and the Elementary School Books on that Subject. By W. G. PECK, Prof. of Mathematics in Columbia College, New York. Price \$1.50.

Davies and Peck's Mathematical Dictionary. 600 pages, 8vo. Price \$2.50.

MATHEMATICAL DICTIONARY, and Cyclopedia of Mathematical Science; comprising Definitions of all the terms employed in Mathematics; An Analysis of each Branch, and of the Whole, as forming a Single Science; By CHARLES DAVIES, L.L.D., Author of a Course of Mathematics; and WILLIAM G. PECK, A.M., Assistant Professor of Mathematics in Columbia College.

*The Publishers of this Work give notice to the Teachers of the United States, who are using DAVIES' COURSE OF MATHEMATICS, in their Schools, — that they shall be furnished with a copy for their own use — for one Dollar — postage to be added, which would be 39 cents. — This offer is open until the first of October, 1859. Address the Publishers,

A. S. BARNES & BURR, 51 and 53 Johns Street, New York.

Philosophical Instruments and Apparatus,

MANUFACTURED AT 313 WASHINGTON STREET, BOSTON,

BY E. S. RITCHIE,

FOR ILLUSTRATING THE SCIENCE OF

Pneumatics, Electricity, Chemistry, Optics, Hydrostatics, Hydraulics,
Steam, Magnetism, Acoustics, Mechanics, Astronomy, Meteorology, &c., with over 600 pieces; also, Mathematical and Engineering Instruments.

Each article is warranted perfect. In designing the form, simplicity, as well as combination of parts in the formation of new Instruments, is carefully studied; and to this end only two sizes of screw connections are used.

RITCHIE'S Illustrated Catalogue, which will be sent by mail on application, contains cuts of over 200 pieces, including his improved Air Pump, Electrical Machine, Ruhmkorff and Atwood Apparatus, &c., the University of Mississippi Electrical Machine, having two six feet diameter plates and four pairs of rubbers; also, sets made up to assist purchasers in selecting, at prices from \$100 to \$1,250 per set; and commendatory letters from eminent Physicists in various parts of the country who are using his apparatus.

WILLIAM BOND AND SON,

17 CONGRESS STREET, BOSTON,

Chronometer Makers to the United States Government,

HAVE FOR SALE A COMPLETE ASSORTMENT

OF MARINE AND SIDEREAL CHRONOMETERS,

ASTRONOMICAL CLOCKS FOR OBSERVATORIES,

WITH THE MOST APPROVED DESCRIPTION OF COMPENSATING PENDULUMS.

They also manufacture the SPRING GOVERNOR APPARATUS, for recording Astronomical Observations by the aid of Electro Magnetism, known as the American Method, and adopted at the Washington and Cambridge Observatories in the United States, and Greenwich and other Observatories in Europe.

Astronomical Instruments, Transits, Telescopes, &c.,

Of the usual description. Also, those for use at fixed Observatories, of larger dimensions, imported to order.